# XV. *On a Class of Invariants.**

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I HAVE not seen it noticed by any mathematician that in the theory of Linear Differential Equations there are two important classes of functions of the coefficients which have remarkable analogies to the invariants of Algebraic Binary Quantics; consequently I venture to call attention to their existence and also to give examples of their application in the present paper.

For convenience I write the equation with binomial coefficients thus

$$\frac{d^ny}{dx^n}+nP_1\frac{d^{n-1}y}{dx^{n-1}}+\frac{n.n-1.}{\lfloor 2}P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_ny=0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where of course $P_1$, $P_2$, &c., are functions of $x$ only.

If now we remove the coefficient of $\frac{d^{n-1}y}{dx^{n-1}}$ by changing $y$ to $ye^{-\int P_1dx}$ the equation, wanting the second term, may be written,

$$\frac{d^ny}{dx^n}+\frac{n.n-1.}{\lfloor 2}H\frac{d^{n-2}y}{dx^{n-2}}+\frac{n.n-1.n-2.}{\lfloor 3}G\frac{d^{n-3}y}{dx^{n-3}}$$

$$+\frac{n.n-1.n-2.n-3.}{\lfloor 4}(I+3H^2)\frac{d^{n-4}y}{dx^{n-4}}+\ldots=0. \quad . \quad . \quad (2)$$

where we have

$$H=P_2-P_1^2-\frac{dP_1}{dx}$$

$$G=2P_1^3-3P_1P_2+P_3-\frac{d^2P_1}{dx^2}$$

$$I=-6P_1^4+12P_1^2P_2-4P_1P_3-3P_2^2+P_4-\frac{d^3P_1}{dx^3}$$

* Since the publication of the abstract of this paper the Rev. R. HARLEY has mentioned to me that the first class of functions treated of here have been already investigated by Sir JAMES COCKLE; having consulted the memoirs I was referred to by Mr. HARLEY, I think little similarity will be found between Sir JAMES COCKLE's results and mine.—J. C. M.

Now H, G, I and the remaining coefficients of equation (2) are in a certain sense invariants of the original equation (1), for they remain unaltered if in equation (1) we change $y$ to $yf(x)$, where $f(x)$ is any function of $x$, and then divide by $f(x)$ so as to make the coefficient of $\frac{d^ny}{dx^n}$ unity; thus writing the equation so transformed

$$\frac{d^ny}{dx^n}+nQ_1\frac{d^{n-1}y}{dx^{n-2}}+\frac{n.n-1.}{\underline{|2}}Q_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots Q_ny=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and for convenience writing $f(x)$, $\frac{df}{dx}$, $\frac{d^2f}{dx^2}$, &c., $f, f', f''$ we find

$$Q_1=\frac{f'+P_1f}{f},\ Q_2=\frac{f''+2P_1f'+P_2f}{f}$$

$$Q_3=\frac{f'''+3P_1f''+3P_2f'+P_3f}{f}$$

Hence also

$$\frac{dQ_1}{dx}=\frac{f''f-f'^2}{f^2}+\frac{dP_1}{dx}$$

$$\frac{d^2Q_1}{dx^2}=\frac{f'''}{f}-\frac{3f'f''}{f^2}+\frac{2f'^3}{f^3}+\frac{d^2P_1}{dx^2}$$

from which we easily prove

$$Q_2-Q_1^2-\frac{dQ_1}{dx}=P_2-P_1^2-\frac{dP_1}{dx}$$

and

$$2Q_1^3-3Q_1Q_2+Q_3-\frac{d^2Q_1}{dx^2}=2P_1^3-3P_1P_2+P_3-\frac{d^2P_1}{dx^2}$$

In a similar manner we find

$$-6Q_1^4+12Q_1^2Q_2-4Q_1Q_3-3Q_2^2+Q_4-\frac{d^3Q_1}{dx^3}=I$$

The theorem proved in these cases may be easily shown to be generally true, as follows:—If from equation (3) we remove the second term by the substitution of $ye^{-\int Q_1dx}$ for $y$ it is evident the result must be identical with (2); but the coefficients of the resulting equation are the same functions of $Q_1$, $Q_2$, $Q_3$, &c., as the coefficients of (2) are of $P_1$, $P_2$, $P_3$, &c., hence the theorem is proved.

I proceed now to some particular applications of the general theorem.

*The quadratic.*

$$\frac{d^2y}{dx^2}+2P_1\frac{dy}{dx}+P_2y=0$$

Here we have the invariant H or

$$P_2-P_1^2-\frac{dP_1}{dx}$$

Let us now seek the condition that the two solutions of the equation $y=y_1$ and $y=y_2$ should be connected by the relation $y_1=y_2x$, which relation, since it depends only on the ratios of $y_1$ and $y_2$, must be expressible in terms of the invariant H.

If in the equation we change $y$ to $yy_2$ the solutions of the resulting equation will be $y=1$ and $y=x$; hence if the equation is

$$\frac{d^2y}{dx^2}+2Q_1\frac{dy}{dx}+Q_2y=0$$

we have $Q_2=0$, $Q_1=0$ remembering now that

$$Q_2-Q_1^2-\frac{dQ_1}{dx}=H$$

the required condition is

$$H=0$$

To solve the equation in this case we have

$$Q_1=\frac{1}{y_2}\left(\frac{dy_2}{dx}+P_1y_2\right)=0$$

Hence $y_2=e^{-\int P_1dx}$ and the complete solution is

$$y=e^{-\int P_1dx}\{Ax+B\}$$

where A and B are arbitrary constants.

We may remark also, as is at once obvious, that the condit of the equations

$$\frac{d^2y}{dx^2}+2P_1\frac{dy}{dx}+P_2y=0$$

and

$$\frac{dy}{dx}+P_1y=0$$

If we seek the mere general condition that the solutions should be connected by the relation $y_1=y_2f(x)$, where $f(x)$ is some given function of $x$, transforming as before by the substitution of $yy_2$ for $y$ the resulting equation

$$\frac{d^2y}{dx^2}+2Q_1\frac{dy}{dx}+Q_2y=0$$

must have for solutions $y=1$ and $y=f(x)$, hence we have

$$Q_2=0 \text{ and } f''(x)+2Q_1f'(x)=0$$

therefore

$$\frac{dQ_1}{dx}+Q_1^2+H=0$$

or substituting the value of $Q_1$ we get

$$2\frac{f'''(x)}{f'(x)}-3\left(\frac{f''(x)}{f'(x)}\right)^2-4H=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the required condition.

To solve the equation in this case, remembering that $Q_1=\frac{1}{y_2}\left(\frac{dy_2}{dx}+P_1y_2\right)$, we at once see that $y_2$ is the solution of the equation

$$2f'(x)\frac{dy}{dx}+\{2P_1f'(x)+f''(x)\}y=0$$

from which

$$y_2=\frac{e^{-\int P_1dx}}{\sqrt{f'(x)}}$$

and the complete solution is

$$y=\frac{e^{-\int P_1dx}}{\sqrt{f'(x)}}\{Af(x)+B\}$$

where A and B are arbitrary constants.

If now in equation (4) we suppose that $f(x)$ is not known, replacing $f(x)$ by $y$ we see the solution of the equation

$$2\frac{d^3y}{dx^3}\cdot\frac{dy}{dx}\cdot-3\left(\frac{d^2y}{dx^2}\right)^2-4H\left(\frac{dy}{dx}\right)^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

is

$$y=\frac{Ay_1+By_2}{Cy_1+Dy_2}$$

where A, B, C, D are arbitrary constants and $y_1$, $y_2$ the solutions of the linear equation

$$\frac{d^2y}{dx^2}+2P_1\frac{dy}{dx}+P_2y=0$$

Again it appears that if $y=\phi(x)$ satisfies equation (5) the complete solution of it is

$$y=\frac{A\phi(x)+B}{C\phi(x)+D}$$

Let us now consider the two equations

$$\frac{d^2y}{dx^2}+2P_1\frac{dy}{dx}+P_2y=0$$

and

$$\frac{d^2y}{dx^2}+2R_1\frac{dy}{dx}+R_2y=0$$

and let us seek the condition that the two solutions of the first equation, $y_1$ and $y_2$, should be connected with the two solutions of the second, $y_3$ and $y_4$, by the relation

$$\frac{y_1}{y_2}=\frac{y_3}{y_4}$$

Let $f(x)$ be the common value of these fractions, then referring to equation (4) we have at once the required condition, viz.:

$$P_2-P_1^2-\frac{dP_1}{dx}=R_2-R_1^2-\frac{dR_1}{dx}$$

It may be remarked that if H does not equal 0 but a constant, say $k$, the complete solution of the equation

$$\frac{d^2y}{dx^2}+2P_1\frac{dy}{dx}+P_2y=0$$

is

$$y=e^{-\int P_1dx}\{A\sin\sqrt{k}x+B\cos\sqrt{k}x\}$$

*The cubic.*

Let us now consider the equation of the third order

$$\frac{d^3y}{dx^3}+3P_1\frac{d^2y}{dx^2}+3P_2\frac{dy}{dx}+P_3y=0$$

we have two invariants G and H; and the equation becomes by removing the second term

$$\frac{d^3y}{dx^3}+3H\frac{dy}{dx}+Gy=0$$

Let us now consider what relation must exist between H and G in order that two solutions of the equation $y_1$ and $y_2$ should be connected by the relation $y_1=y_2x$.

Transform the cubic by substituting $yy_2$ for $y$ and let the resulting equation be

$$\frac{d^3y}{dx^3}+3Q_1\frac{d^2y}{dx^2}+3Q_2\frac{dy}{dx}+Q_3y=0$$

Since this equation is satisfied by $y=1$ and also by $y=x$ we have at once

$$Q_3=0 \text{ and } Q_2=0$$

hence

$$-Q_1^2-\frac{dQ_1}{dx}=H$$

$$2Q_1^3-\frac{d^2Q_1}{dx^2}=G$$

from which we find

$$G-\frac{dH}{dx}+2HQ_1=0$$

Substituting the value of $Q_1$ derived from this equation in H and reducing we find for our required condition

$$4H^3+G^2-\left(\frac{dH}{dx}\right)^2+2H\left(\frac{d^2H}{dx^2}-\frac{dG}{dx}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This condition, as is seen, is the eliminant of the equations

$$\frac{d^3y}{dx^3}+3P_1\frac{d^2y}{dx^2}+3P_2\frac{dy}{dx}+P_3y=0$$

and

$$\frac{d^2y}{dx^2}+2P_1\frac{dy}{dx}+P_2y \qquad =0$$

To solve the equation when condition (6) is true. First consider the equation with the second term removed, and we have to find $y_2$ the equations

$$\frac{d^3y}{dx^3}+3H\frac{dy}{dx}+Gy=0$$

$$\frac{d^2y}{dx^2}+Hy \qquad =0$$

of which equations $y_2$ must be a common solution, as follows at once from the conditions $Q_3=0$, $Q_2=0$. Thence we easily find

$$y_2=\sqrt{H}e^{-\frac{1}{2}\int\frac{G}{H}dx}$$

and therefore

$$y_1=x\sqrt{H}e^{-\frac{1}{2}\int\frac{G}{H}dx}$$

These values of $y_1$ and $y_2$ are to be multiplied by $e^{-\int P_1dx}$ to get the corresponding solutions of the cubic when the second term is not removed.

To get the third solution we have

$$\frac{d^3y}{dx^3}+3Q_1\frac{d^2y}{dx^2}=0$$

which gives, remembering the relation

$$G-\frac{dH}{dx}+2HQ_1=0$$

$$y=\iint H^{\frac{3}{2}}e^{-\frac{3}{2}\int\frac{G}{H}dx}dx^2$$

thence we easily find the complete solution of the original cubic

$$y=\sqrt{H}e^{-\int\left(P_1+\frac{G}{2H}\right)dx}\left\{A+Bx+C\iint H^{\frac{3}{2}}e^{-\frac{3}{2}\int\frac{G}{H}dx}dx^2\right\}$$

where A, B, and C are arbitrary constants.

If we have the two conditions $G=0$ and $H=0$, then

$$y_1=y_2x=y_3x^2$$

and the complete solution is

$$y=e^{-\int P_1dx}\{A+Bx+Cx^2\}$$

$G=0$ expresses the condition that $y=e^{-\int P_1dx}$ should be a solution of the equation.

If we wish to find the conditions necessary that the following relations should exist between the solutions of the cubic

$$y_1=y_3\phi(x), \qquad y_2=y_3\psi(x)$$

where $\phi(x)$ and $\psi(x)$ are known functions of $x$; change as before the cubic by substituting $yy_3$ for $y$ and let it become

$$\frac{d^3y}{dx^3}+3Q_1\frac{d^2y}{dx^2}+3Q_2\frac{dy}{dx}+Q_3y=0$$

we must evidently have $Q_3=0$, hence we have

$$\phi'''+3Q_1\phi''+3Q_2\phi'=0$$

$$\psi'''+3Q_1\psi''+3Q_2\psi'=0$$

we have also

$$Q_2-Q_1^2-\frac{dQ_1}{dx}=H$$

$$Q_1^3-3Q_1Q_2-\frac{d^2Q_1}{dx^2}=G$$

Now if we let

$$\frac{\phi'''\psi'-\psi'''\phi'}{3(\phi''\psi'-\psi''\phi')}\equiv-F(x)$$

and

$$\frac{\phi'''\psi''-\psi'''\phi''}{3(\phi''\psi'-\psi''\phi')}\equiv f(x)$$

we have $Q_1=F(x)$, $Q_2=f(x)$, and the conditions sought are

$$H+F'(x)+(F(x))^2-f(x)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$G+F''(x)+3F(x)f(x)-(F(x))^3=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

To solve the equation in this case we have

$$Q_1=\frac{1}{y_3}\left(\frac{dy_3}{dx}+P_1y_3\right)$$

thence $y_3$ must be the solution of the equation

$$\frac{dy}{dx}+\{P_1-F(x)\}y=0$$

from which we have

$$y=e^{\int\{F(x)-P_1\}dx}$$

or

$$y=\frac{e^{-\int P_1 dx}}{(\phi''\psi'-\psi''\phi')^{\frac{1}{3}}}$$

and therefore the complete solution of the cubic is

$$y=\frac{e^{-\int P_1 dx}}{(\phi''\psi'-\psi''\phi')^{\frac{1}{3}}}\{A+B\phi(x)+C\psi(x)\}$$

If we make $\phi(x)=\frac{1}{\psi(x)}$ and then eliminate $\psi(x)$ between equations (7) and (8) we should evidently obtain the condition necessary that the relation $y_1y_2=y_3^2$ should exist. I have, however, obtained the condition in a much more simple manner, and found it to be

$$2G-3\frac{dH}{dx}=0$$

which result I give further on in the present paper.

Let us now consider the more general problem; to determine the relation between the coefficients of the cubic in order that two solutions should be connected by the relation $y_1=y_2f(x)$ where $f(x)$ is a given function of $x$.

As before, change $y$ to $y_2y$, and the equation must become of the form

$$\frac{d^3y}{dx^3}+3Q_1\frac{d^2y}{dx^2}+3Q_2\frac{dy}{dx}=0$$

$Q_3$ vanishing.

We have then

$$f'''+3Q_1f''+3Q_2f'=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Also

$$Q_2-Q_1^2-\frac{dQ_1}{dx}=H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$2Q_1^3-3Q_1Q_2-\frac{d^2Q_1}{dx^2}=G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and the problem is to eliminate $Q_1$ and $Q_2$ from these equations.

If we substitute in (10) and (11) the value of $Q_2$ found from (9), we find they can be written in the forms

$$-R^2-\frac{dR}{dx}=S$$

$$2R^3+RK-\frac{d^2R}{dx^2}=T$$

where

$$R=Q_1+\frac{f''}{2f'},\quad S=H-\frac{f'''}{6f'}+\frac{f''^2}{4f'^2}$$

$$K=\frac{2f'f'''-3f''^2}{2f'^2},\quad T=G-\frac{f''^3}{f'^3}+\frac{2f''f'''}{f'^2}-\frac{f^{(iv)}}{2f'}$$

we have

$$\frac{dS}{dx}-T=(2S-K)R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Substituting this value of R in S and reducing we find for the required condition

$$4S^3+T^2-\left(\frac{dS}{dx}\right)^2+2S\left(\frac{d^2S}{dx^2}-\frac{dT}{dx}\right)+SK^2-LS^2K+K\left(\frac{dT}{dx}-\frac{d^2S}{dx^2}\right)-\frac{dK}{dx}\left(T-\frac{dS}{dx}\right)=0\ . \quad (13)$$

If K vanishes or $f'^2=Af''^3$, A being a constant, equation (13) is derived from (6) by changing H and G to S and T respectively.

If we regard $f(x)$ as unknown, equal $u$ say, equation (12) is the differential equation of the fifth order, of which the complete solution is

$$u=\frac{Ay_1+By_2+Cy_3}{Dy_1+Ey_2+Fy_3}$$

where $y_1$, $y_2$, $y_3$ are solutions of the equation

$$\frac{d^3y}{dx^3}+3P_1\frac{d^2y}{dx^2}+3P_2\frac{dy}{dx}+P_3y=0$$

and A, B, C, D, E, F arbitrary constants.

To solve the cubic when condition (13) holds. From equations (9) and 12 we find at once $Q_1$ and $Q_2$ in terms of $x$, let their values be respectively $\phi(x)$ and $\psi(x)$ we have then

$$\phi(x)=\frac{\frac{dS}{dx}-T}{2S-K}-\frac{f''}{2f'}$$

$$\psi(x)=\frac{f''^2}{2f'^2}-\frac{\frac{dS}{dx}-T}{2S-K}\cdot\frac{f''}{f'}-\frac{f'''}{3f'}$$

Now since

$$Q_1=\frac{1}{y_2}\left(\frac{dy_2}{dx}+P_1y_2\right)$$

we have

$$y_2=e^{\int(\phi(x)-P_1)dx}$$

therefore

$$y_1=f(x)e^{\int(\phi(x)-P_1)dx}$$

To determine $y_3$, let $y_3=y_2\chi(x)$ and we have to determine $\chi$

$$\chi'''+3Q_1\chi''+3Q_2\chi'=0$$

but also

$$f'''+3Q_1f''+3Q_2f'=0$$

from which we easily find

$$\chi''f'-\chi'f''=e^{3\int Q_2dx}=e^{3\int\psi(x}$$

therefore

$$\chi=\int\left\{f'\int\frac{1}{f'^2}e^{3\int\psi(x)dx}dx\right\}dx$$

and the complete solution of the cubic is in the case we are considering.

$$y=e^{\int(\phi(x)-P_1)dx}\int\left\{f'\int\frac{1}{f'^2}\,e^{3\int\psi(x)dx}dx\right\}dx$$

remembering that an arbitrary constant is implied in each integration.

## *The quartic.*

Consider now the equation of the fourth order

$$\frac{d^4y}{dx^4}+4P_1\frac{d^3y}{dx^3}+6P_2\frac{d^2y}{dx^2}+4P_3\frac{dy}{dx}+P_4y=0$$

we have three invariants H, G, and I, and the equation with its second term removed becomes

$$\frac{d^4y}{dx^4}+6H\frac{d^2y}{dx^2}+4G\frac{dy}{dx}+(I+3H^2)y=0$$

If we seek the condition that two solutions of the equation should be connected by the relation $y_1=y_2x$, transforming the equation by changing $y$ to $yy_2$, the result must be of the form

$$\frac{d^4y}{dx^4}+4Q_1\frac{d^3y}{dx^3}+6Q_2\frac{d^2y}{dx^2}=0$$

Hence we have

$$H=Q_2-Q_1^2-\frac{dQ_1}{dx}$$

$$G=2Q_1^3-3Q_1Q_2-\frac{d^2Q_1}{dx^2}$$

$$I=-6Q_1^4+12Q_1^2Q_2-3Q_2^2-\frac{d^3Q_1}{dx^3}$$

and the condition required is the result of eliminating $Q_1$ and $Q_2$ from these equations.

The required result is also the eliminant of the equations

$$\frac{d^4y}{dx^4}+6\mathrm{H}\frac{d^2y}{dx^2}+4\mathrm{G}\frac{dy}{dx}+(\mathrm{I}+3\mathrm{H}^2)y=0$$

and

$$\frac{d^3y}{dx^3}+3\mathrm{H}\frac{dy}{dx}+\mathrm{G}y=0$$

as is evident.

Having obtained the result in a very cumbersome form only I do not give it, but at once go on to cases where two or three conditions subsist between the solutions of the quartic.

Let us first consider the case where

$$y_1=y_2x=y_3x^2$$

changing $y$ to $yy_3$ the resulting equation must be of the form

$$\frac{d^4y}{dx^4}+4\mathrm{Q}_1\frac{d^3y}{dx^3}=0$$

Hence we have

$$\mathrm{H}=-\mathrm{Q}_1^2-\frac{d\mathrm{Q}_1}{dx},\ \mathrm{G}=2\mathrm{Q}_1^3-\frac{d^2\mathrm{Q}_1}{dx^2}$$

$$\mathrm{I}=-6\mathrm{Q}_1^4-\frac{d^3\mathrm{Q}_1}{dx^3}$$

from which

$$\mathrm{G}-\frac{d\mathrm{H}}{dx}+2\mathrm{Q}_1\mathrm{H}=0,\ \mathrm{I}-\frac{d\mathrm{G}}{dx}=6\mathrm{Q}_1^2\mathrm{H}$$

one of the required conditions is then given by equation (6), and the second is easily found to be

$$2\mathrm{H}\left(\mathrm{I}-\frac{d\mathrm{G}}{dx}\right)=3\left(\mathrm{G}-\frac{d\mathrm{H}}{dx}\right)^2$$

To solve the equation, we find at once as in the case of the equation of the third order

$$y_3=e^{-\int \mathrm{P}_1 dx}\sqrt{\mathrm{H}}e^{-\frac{1}{2}\int\frac{\mathrm{G}}{\mathrm{H}}dx}$$

and $y_4$ is the solution of the equation

$$\frac{d^4y}{dx^4}+4\mathrm{Q}_1\frac{d^3y}{dx^3}=0$$

multiplied by $y_3$.

Hence remembering the value of $\mathrm{Q}_1$, we get for the complete solution of the quartic

$$y=\mathrm{A}\sqrt{\mathrm{H}}e^{-\int\left(\mathrm{P}_1+\frac{\mathrm{G}}{2\mathrm{H}}\right)dx}\left\{\iiint \mathrm{H}^2e^{-2\int\frac{\mathrm{G}}{\mathrm{H}}dx}\,dx^3+\mathrm{B}x^2+\mathrm{C}x+\mathrm{D}\right\}$$

If $y_1=y_2x=y_3x^2=y_4x^3$ then we have

$$H=0,\ G=0,\ I=0$$

and the complete solution is

$$y=e^{-\int P_1dx}(Ax^3+Bx^2+Cx+D)$$

It is obvious that when H, G, and I instead of being each equal to 0 are each equal to a given constant the equation can be at once solved.

If we seek the conditions for

$$y_1=y_4\phi(x),\ y_2=y_4\psi(x),\ y_3=y_4\chi(x)$$

where $\phi(x)$, $\psi(x)$, $\chi(x)$ are given functions of $x$; then changing $y$ to $yy_4$ if the resulting equation be written

$$\frac{d^4y}{dx^4}+4Q_1\frac{d^3y}{dx^3}+6Q_2\frac{d^2y}{dx^2}+4Q_3\frac{dy}{dx}=0$$

we have

$$\phi^{(iv)}+4Q_1\phi'''+6Q_2\phi''+4Q_3\phi'=0$$
$$\psi^{(iv)}+4Q_1\psi'''+6Q_2\psi''+4Q_3\psi'=0$$
$$\chi^{(iv)}+4Q_1\chi'''+6Q_2\chi''+4Q_3\chi'=0$$

from which equations we find $Q_1$, $Q_2$, $Q_3$ are terms of $x$. Let these values be $F_1(x)$, $F_2(x)$, and $F_3(x)$, and the required conditions are

$$H=F_2(x)-(F_1(x))^2-F_1'(x)$$
$$G=2(F_1(x))^3-3F_1(x)F_2(x)+2F_3(x)-F_1''(x)$$
$$I=-6(F_1(x))^4+12(F_1(x))^2F_2(x)-4F_1(x)F_3(x)-3(F_2(x))^2-F_1'''(x)$$

To find the solution in this case, we have

$$Q_1y_4=\frac{dy_4}{dx}+P_1y_4$$

Hence

$$y_4=e^{\int(Q_1-P_1)dx}$$

Now if we let

$$\begin{vmatrix} \phi' & \psi' & \chi' \\ \phi'' & \psi'' & \chi'' \\ \phi''' & \psi''' & \chi''' \end{vmatrix}=\Delta$$

we find from the previous equations

$$Q_1=-\frac{\frac{d\Delta}{dx}}{4\Delta}$$

and therefore we have for the complete solution

$$y=\frac{e^{-\int P_1 dx}}{\Delta^{\frac{1}{4}}}\{A\phi(x)+B\psi(x)+C\chi(x)+D\}$$

If

$$\phi(x)=e^{\alpha x},\quad \psi(x)=e^{\beta x},\quad \chi(x)=e^{\gamma x},$$

$Q_1$, $Q_2$, $Q_3$ are constants determined from the equations previously given; in fact we easily find

$$4Q_1=-(\alpha+\beta+\gamma),\quad 6Q_2=\alpha\beta+\alpha\gamma+\beta\gamma,\quad 4Q_3=-\alpha\beta\gamma$$

H, G, and I are constants, and the solution is

$$y=e^{-\frac{\alpha+\beta+\gamma}{4}x-\int P_1 dx}\{Ae^{\alpha x}+Be^{\beta x}+Ce^{\gamma x}+D\}$$

## II.

In addition to the class of invariants of Linear Differential Equations which I have discussed in the first part of this paper, there is also another class worth noticing, namely, functions of the coefficients of the equation which remain unaltered when the *independent* variable is changed. I propose now to consider these functions.

### *The quadratic.*

If we take the equation

$$\frac{d^2y}{dx^2}+2P_1\frac{dy}{dx}+P_2 y=0 \quad \ldots \ldots \ldots \quad (14)$$

and let $x=\phi(t)$, the new equation is

$$\frac{d^2y}{dt^2}+2Q_1\frac{dy}{dt}+Q_2 y=0$$

where

$$2Q_1=2P_1\phi'-\frac{\phi''}{\phi'},\quad Q_2=P_2\phi'^2$$

From these values of $Q_1$ and $Q_2$ we easily find the identity

$$\frac{\frac{dQ_2}{dt}+4Q_1Q_2}{Q_2^{\frac{3}{2}}}\equiv\frac{\frac{dP_2}{dx}+4P_1P_2}{P_2^{\frac{3}{2}}}$$

Hence we see that

$$\frac{\frac{dP_2}{dx}+4P_1P_2}{P_2^{\frac{3}{2}}}$$

is an invariant of this kind of the equation of the second order.

To arrive at this invariant directly, let us suppose that the second term is removed from the equation by the change of the independent variable; we have then, supposing the transformed equation to be

$$\frac{d^2y}{dt^2}+uy=0$$

$$2P_1\phi'-\frac{\phi''}{\phi'}=0, \quad u=P_2\phi'^2$$

Hence

$$\frac{du}{dt}=\phi'^3\frac{dP_2}{dx}+2P_2\phi'\phi''=\phi'^3\left(\frac{dP_2}{dx}+4P_1P_2\right)$$

therefore

$$\frac{\frac{dP_2}{dx}+4P_1P_2}{P_2^{\frac{3}{2}}}=\frac{1}{u^{\frac{3}{2}}}\frac{du}{dt}$$

from which it is evident that

$$\frac{\frac{dP_2}{dx}+4P_1P_2}{P_2^{\frac{3}{2}}}$$

remains the same when the independent variable is changed.

I shall denote this invariant by the letter J; and I now propose to give some examples of its use.

Let us seek to determine what relation among the coefficients of the equation expresses the condition that the two solutions of equation (14) $y_1$ and $y_2$ should be connected by the equation $y_1y_2=1$.

Transform the independent variable so that $e^t$ shall be a solution of the new equation, then $e^{-t}$ must also be a solution. Let the new equation be

$$\frac{d^2y}{dt^2}+2Q_1\frac{dy}{dt}+Q_2y=0$$

and we have

$$1+2Q_1+Q_2=0$$
$$1-2Q_1+Q_2=0$$

from which we find $Q_1=0$, $Q_2=-1$, but

$$J=\frac{\frac{dQ_2}{dt}+4Q_1Q_2}{Q_2^{\frac{3}{2}}}=0$$

hence the required condition is $J=0$ or

$$\frac{dP_2}{dx}+4P_1P_2=0$$

to solve the equation in this case, we have

$$Q_2=P_2\phi'^2 \quad \text{or} \quad 1+P_2\frac{dx^2}{dt^2}=0$$

from which

$$t=e\int\sqrt{-P_2}dx$$

and the complete solution is

$$y=\mathrm{A}e^{\int\sqrt{-P_2}dx}+\mathrm{B}e^{-\int\sqrt{-P_2}dx}$$

where A and B are arbitrary constants.

Again, let us seek the condition for $y_1y_2^2=1$, transforming as before so that $y_1=e^t$, we find now that the coefficients of the transformed equation are connected by the relations

$$1+2Q_1+Q_2=0$$
$$4-4Q_1+Q_2=0$$

therefore

$$2Q_1=1,\ Q_2=-2$$

and we have

$$J^2+2=0$$

or

$$\left(\frac{dP_2}{dx}+4P_1P_2\right)^2+2P_2^3=0$$

for the required condition.

In this case we find as before the solution to be

$$y=\mathrm{A}e^{\frac{1}{\sqrt{2}}\int\sqrt{-P_2}dx}+\mathrm{B}e^{-\sqrt{2}\int\sqrt{-P_2}dx}$$

More generally if $y_1=y_2^m$ we have, using the same transformation as before, the following equations connecting the coefficients of the transformed equation

$$1+2Q_1\ \ +Q_2=0$$
$$m^2+2mQ_1+Q_2=0$$

from which

$$2Q_1=-(m+1), \quad Q_2=m$$

and we find as before for the required condition

$$m\left(\frac{dP_2}{dx}+4P_1P_2\right)^2-4(m+1)^2P_2^3=0$$

the solution being

$$y=\mathrm{A}e^{\frac{1}{\sqrt{m}}\int\sqrt{P_2}dx}+\mathrm{B}e^{\sqrt{m}\int\sqrt{P_2}dx}$$

this investigation fails, as it should, when $m=1$.

As another example let us suppose $y_1$ and $y_2$ connected by any relation; for convenience take $y_2=f(\log y_1)$ and let us seek to determine how the coefficients of the equation are related, and also its solution.

Transforming as before we have

$$1+2Q_1 \quad +Q_2 \quad =0$$
$$f''(t)+2f'(t)Q_1+Q_2f(t)=0$$

from which

$$2Q_1=\frac{f(t)-f''(t)}{f'(t)-f(t)}$$
$$Q_2=\frac{f'(t)-f''(t)}{f(t)-f'(t)}$$

from this we have

$$\int\sqrt{P_2}dx=\int\sqrt{\frac{f'(t)-f''(t)}{f(t)-f'(t)}}dt=F(t) \text{ say}$$

hence

$$t=F^{-1}\left\{\int\sqrt{P_2}dx\right\}$$

and the complete solution of the equation is

$$y=Ae^{F^{-1}\{\int\sqrt{P_2}dx\}}+Bf\{F^{-1}\{\int\sqrt{P_2}dx\}\}$$

to determine the condition between $P_1$ and $P_2$ we have

$$\frac{\frac{dQ_1}{dt}+4Q_1Q_2}{Q_2^{\frac{3}{2}}}=J$$

substituting $F^{-1}\{\int\sqrt{P_2}dx\}$ for $t$ in the left-hand side of this equation we get the required result.

It is to be remarked that $\int\sqrt{P_2}dx$ is an invariant, since if the substitution $x=\phi(t)$ removes the second term from the equation we have

$$\int\sqrt{P_2}dx=\int\sqrt{u}dt$$

where $u$ has the same meaning as before.

From the equation $J=0$ which expresses the condition $y_1y_2=1$ we can derive the linear differential equation of the third order, of which the solution is

$$y=Ay_1^2+By_2^2+Cy_1y_2$$

where A, B, and C are arbitrary constants.

As I afterwards make use of this equation I shall here give a full investigation of it.

The condition $J=0$, it is easily seen may be written

$$\frac{d^2P_1}{dx^2}+6P_1\frac{dP_1}{dx}+4P_1^3+4P_1H+\frac{dH}{dx}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where H is the invariant of the first kind previously considered.

If now in the equation

$$\frac{d^2y}{dx^2}+2P_1\frac{dy}{dx}+P_2y=0$$

we change $y$ to $y\sqrt{z}$ we find the equation

$$\frac{d^2y}{dx^2}+2Q_1\frac{dy}{dx}+Q_2y=0$$

where

$$2Q_1=2P_1+\frac{1}{z}\frac{dz}{dx}$$

Substituting this value of $Q_1$ for $P_1$ in condition (15) and reducing we get

$$\frac{d^3z}{dx^3}+6P_1\frac{d^2z}{dx^2}+2\left(\frac{dP_1}{dx}+4P_1^2+2P_2\right)\frac{dz}{dx}+2\left(4P_1P_2+\frac{dP_2}{dx}\right)z=0 \quad . \quad . \quad . \quad . \quad (A)$$

this differential equation in $z$ is evidently the required equation.

It is to be remarked that if we remove the second term from this equation it becomes

$$\frac{d^3z}{dx^3}+4H\frac{dz}{dx}+2\frac{dH}{dx}z=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

*The cubic.*

Let us now consider the equation of the third order

$$\frac{d^3y}{dx^3}+3P_1\frac{d^2y}{dx^2}+3P_2\frac{dy}{dx}+P_3y=0$$

Substituting $\phi(t)$ for $x$ we get

$$\frac{d^3y}{dt^3}+3Q_1\frac{d^2y}{dt^2}+3Q_2\frac{dy}{dt}+Q_3y=0$$

where

$$3Q_1=3P_1\phi'-\frac{3\phi''}{\phi'}, \quad Q_3=P_3\phi'^3$$

$$3Q_2=2P_2\phi'^2-\frac{\phi'''}{\phi'}+\frac{3\phi''^2}{\phi'^2}-3P_1\phi''$$

Now let

$$Q_1=0,\ 3Q_2=u,\ Q_3=v$$

from which

$$P_1=\frac{\phi''}{\phi'^2},\ u=3P_2\phi'^2-\frac{\phi''}{\phi'},\ v=P_3\phi'^3$$

from which equations two distinct invariants, of the second kind, of the cubic may be found, thus

$$\frac{dv}{dt}=\phi'^4\frac{dP_3}{dx}+3\phi'^2\phi''P_3=\phi'^4\left(\frac{dP_3}{dx}+3P_1P_3\right)$$

therefore

$$\frac{\frac{dP_3}{dx}+3P_1P_3}{P_3^{\frac{4}{3}}}=\frac{1}{v^{\frac{4}{3}}}\frac{dv}{dt}$$

Hence

$$\frac{\frac{dP_3}{dx}+3P_1P_3}{P_3^{\frac{4}{3}}}$$

is an invariant which I shall call $I_1$.

Again

$$\frac{dP_1}{dx}=-\frac{2\phi''^2}{\phi'^4}+\frac{\phi''}{\phi'^3}$$

or

$$\phi'^2\left\{\frac{dP_1}{dx}+2P_1^2-3P_2\right\}=-u$$

Hence we have

$$\frac{\frac{dP_1}{dx}+2P_1^2-3P_2}{P_3^{\frac{2}{3}}}=-\frac{u}{v^{\frac{2}{3}}}$$

and we have another invariant

$$\frac{\frac{dP_1}{dx}+2P_1^2-3P_2}{P_3^{\frac{2}{3}}}$$

which I shall call $I_2$.

We have also, calling $-\frac{dP_1}{dx}-2P_1^2+3P_2$, L,

$$\phi'^2L=u$$

therefore

$$\phi'^3\frac{dL}{dx}+2\phi'\phi''L=\frac{du}{dt}$$

from which

$$\frac{\frac{dL}{dx}+2P_1L}{L^{\frac{3}{2}}}=\frac{1}{u^{\frac{3}{2}}}\frac{du}{dt}$$

and we have the invariant

$$\frac{\frac{dL}{dx}+2P_1L}{L^{\frac{3}{2}}},$$

$I_3$ say.

$I_3$, however, is not a distinct invariant, since it is evident from the method of forming it that it can be expressed in terms of $I_1$ and $I_2$.

It is to be remarked that if I is any invariant of the cubic, of the kind we are considering, then

$$\frac{1}{P_3^{\frac{1}{3}}}\frac{dI}{dx} \quad \text{and} \quad \int P_3^{\frac{1}{3}} I dx$$

are also invariants of the same kind, as follows at once from the relation

$$P_3\phi'^3 = v \quad \text{or} \quad P_3^{\frac{1}{3}}dx = v^{\frac{1}{3}}dt.$$

Let us now seek the condition that two solutions of the cubic, $y_1$, $y_2$ should be connected by the relation $y_1y_2=1$.

Transform the equation so that $e^t$ shall be a solution and let it become

$$\frac{d^3y}{dt^3}+3Q_1\frac{d^2y}{dt^2}+3Q_2\frac{dy}{dt}+Q_3y=0$$

and we have

$$1+3Q_1+3Q_2+Q_3=0$$

$$1-3Q_1+3Q_2-Q_3=0$$

from which

$$1+3Q_2=0, \quad 3Q_1+Q_3=0$$

Substituting for $Q_2$ and $Q_1$ in the equations

$$\frac{\frac{dQ_3}{dt}+3Q_1Q_3}{Q_3^{\frac{4}{3}}}=I_1$$

$$\frac{\frac{dQ_1}{dt}+2Q_1^2-3Q_2}{Q_3^{\frac{2}{3}}}=I_2$$

then letting $Q_3=z^3$ and reducing we get

$$3\frac{dz}{dt}=I_1z^2+z^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$9z^2\frac{dz}{dt}=2z^6-9I_2z^2+9$$

from which equations we are to eliminate $z$. Eliminating $\frac{dz}{dt}$, we get

$$z^6+3I_1z^4+9I_2z^2-9=0 \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now differentiating with regard to $t$ and then substituting for $\frac{dz}{dt}$ from (16) we find

$$2z^6+4I_1z^4+z^2\left(4I_1^2-6I_2+\frac{3}{P_3^{\frac{1}{3}}}\frac{dI_1}{dx}\right)-6I_1I_2+\frac{9}{P_3^{\frac{1}{3}}}\frac{dI_2}{dx}=0 \quad . \quad . \quad . \quad (18)$$

Hence the required result is the eliminant of the two cubic equations (17) and (18).

I do not give here the expanded result thus obtained, as I have arrived at it in a more compact form, as follows :

Remove the second term, from the cubic we are considering, by change of the independent variable, and let the result be

$$\frac{d^3y}{dt^3}+u\frac{dy}{dt}+vy=0$$

Suppose now that $y_1y_2=1$, and let

$$\frac{d^2y}{dt^2}+2\mathrm{Q}_1\frac{dy}{dt}+\mathrm{Q}_2y=0$$

be the linear equation of which the solutions are $y_1$ and $y_2$, it is evident then that the cubic may be written in the form

$$\frac{d}{dt}\left\{\frac{d^2y}{dt^2}+2\mathrm{Q}_1\frac{dy}{dt}+\mathrm{Q}_2y\right\}-2\lambda\left\{\frac{d^2y}{dt^2}+2\mathrm{Q}_1\frac{dy}{dt}+\mathrm{Q}_2y\right\}=0$$

Since this equation is evidently satisfied by $y_1$ and $y_2$, and we can determine $\lambda$ so that any other function of $t$ shall satisfy it.

Comparing coefficients we find

$$2\mathrm{Q}_1-2\lambda=0,\quad 2\frac{d\mathrm{Q}_1}{dt}-2\lambda\mathrm{Q}_1+\mathrm{Q}_2=u$$

$$\frac{d\mathrm{Q}_2}{dt}-2\lambda\mathrm{Q}_2=v$$

we have also in consequence of the given condition

$$\frac{d\mathrm{Q}_2}{dt}+4\mathrm{Q}_1\mathrm{Q}_2=0$$

From these we have

$$6\mathrm{Q}_2\lambda+v=0\quad 3\mathrm{Q}_2+2\int vdt=0$$

$$2\frac{d\lambda}{dt}-2\lambda^2+\mathrm{Q}_2=u$$

Hence we easily derive

$$16\left\{\int vdt\right\}^3+24u\left\{\int vdt\right\}^2-12\frac{dv}{dt}\int vdt+15v^2=0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

which is the required result expressed in terms of the invariants $u$ and $v$.

To write the result in terms of $\mathrm{I}_1$ and $\mathrm{I}_2$, we have $\mathrm{I}_1=\frac{1}{v^{\frac{4}{3}}}\frac{dv}{dt}$ from which we get

$$v=e^{\int \mathrm{P}_3^{\frac{1}{3}}\mathrm{I}_1dx}$$

and therefore

$$\int v dt = \int P^{\frac{1}{3}}_3 e^{\frac{2}{3}\int P_3^{\frac{1}{3}} I_1 dx}$$

$$\frac{dv}{dt} = I_1 e^{\frac{4}{3}\int I_1 P_3^{\frac{1}{3}} dx} \quad u = -I_2 e^{\frac{2}{3}\int P_3^{\frac{1}{3}} I_1 dx}$$

Hence letting

$$S = e^{\int P_3^{\frac{1}{3}} I_1 dx}, \quad R = \int P^{\frac{1}{3}}_3 e^{\frac{2}{3}\int I_1 P_3^{\frac{1}{3}} dx}$$

Condition (19) may be written in terms of the coefficients of the cubic without its second term being removed as follows:

$$16R^3 - 24I_2 R^2 S^{\frac{2}{3}} - 12I_1 R S^{\frac{4}{3}} + 15S^2 = 0$$

Having expressed $Q_2$ and $v$ as functions of $x$ as above, the cubic may in this case be solved as follows:

We have from previous results the solutions of

$$\frac{d^2y}{dt^2} + Q_1 \frac{dy}{dt} + Q_2 y = 0$$

where $Q_1$ and $Q_2$ are connected by the relation

$$\frac{dQ_2}{dt} + 4Q_1 Q_2 = 0$$

$$y_1 = e^{\int \sqrt{-Q_2} dt}, \quad y_2 = e^{-\int \sqrt{-Q_2} dt}$$

Hence two solutions of the cubic are

$$y_1 = e^{\int \frac{\sqrt{-Q_2}}{v^{\frac{1}{2}}} P_3^{\frac{1}{3}} dx}, \quad y_2 = e^{\int \frac{\sqrt{-Q_2}}{v^{\frac{1}{2}}} P_3^{\frac{1}{3}} dx}$$

and the complete solution may then be found.

If we seek the conditions for

$$y_1 = y_2^m = y_3^n$$

transform so that $y_1$ may become $e^t$, and as before we have

$$\begin{aligned} 1 + 3Q_1 \quad + 3Q_2 \quad + Q_3 &= 0 \\ m^3 + 3Q_1 m^2 + 3Q_2 m + Q_3 &= 0 \\ n^3 + 3Q_1 n^2 + 3Q_2 n + Q_3 &= 0 \end{aligned}$$

from which we find $Q_1$, $Q_2$, $Q_3$ three constants $\alpha$, $\beta$, $\gamma$, say. Hence the required conditions are

$$\frac{3\alpha}{\gamma^{\frac{4}{3}}} = I_1 \quad \text{and} \quad \frac{2\alpha^2 - 3\beta}{\gamma^{\frac{2}{3}}} = I_2$$

and the solution is easily found to be

$$y = Ae^{\int \left(\frac{P_3}{\gamma}\right)^{\frac{1}{3}} dx} + Be^{m\int \left(\frac{P_3}{\gamma}\right)^{\frac{1}{3}} dx} + Ce^{n\int \left(\frac{P_3}{\gamma}\right)^{\frac{1}{3}} dx}$$

More generally let us seek the conditions that $y_2$ and $y_3$ should be any given functions of $y_1$.

Write for convenience $y_2=\phi(\log y_1)$, $y_3=\psi(\log y_1)$ then transforming as before we have

$$\begin{aligned} 1\ \ +3Q_1\ \ +3Q_2\ \ +Q_3&=0 \\ \phi'''+3Q_1\phi''+3Q_2\phi'+Q_3\phi&=0 \\ \psi'''+3Q_1\psi''+3Q_2\psi'+Q_3\psi&=0 \end{aligned}$$

Now from these equations we find

$$Q_3=-\frac{\begin{vmatrix} 1 & 1 & 1 \\ \phi''' & \phi'' & \phi' \\ \psi''' & \psi'' & \psi' \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ \phi & \phi'' & \phi' \\ \psi & \psi'' & \psi' \end{vmatrix}}$$

Hence from the relation $\int P_3^{\frac{1}{3}}dx=\int Q_3^{\frac{1}{3}}dt$ we find $\int P_3^{\frac{1}{3}}dx=F(t)$ where F is a known function, and the complete solution of the cubic is

$$y=Ae^{F^{-1}\{\int P_3^{\frac{1}{3}}dx\}}+B\phi\left\{F^{-1}\left\{\int P_3^{\frac{1}{3}}dx\right\}\right\}+C\psi\left\{F^{-1}\left\{\int P_3^{\frac{1}{3}}dx\right\}\right\}$$

To find the required conditions we have only to substitute $F^{-1}\{\int P_3^{\frac{1}{3}}dx\}$ for $t$ in the left-hand sides of equations

$$\frac{\frac{dQ_3}{dt}+3Q_1Q_3}{Q_3^{\frac{4}{3}}}=I_1, \quad \frac{\frac{dQ_1}{dt}+2Q_1^2-3Q_2}{Q_3^{\frac{2}{3}}}=I_2$$

$Q_1$, $Q_2$, $Q_3$ being found in terms of $t$ from equations previously given.

I proceed now to consider an invariant of the cubic which is particularly worth noticing.

Referring to the values of $I_2$ and $I_3$ given before, we find

$$I_3-\frac{2}{(-I_2)^{\frac{3}{2}}}\equiv\frac{\frac{dL}{dx}+2P_1L-2P_3}{L^{\frac{3}{2}}}=K \quad \text{(say)}$$

and $K=0$ is the condition that the solutions of the equation of the third order should be connected by the relation $y_3^2=y_1y_2$; as follows.

Let $y_1=z_1^2$, $y_2=z_2^2$, and let the equation of which the solutions are $z_1$ and $z_2$ be

$$\frac{d^2z}{dx^2}+2Q_1\frac{dz}{dx}+Q_2z=0$$

Hence referring to equation (A) we see that

$$\frac{d^3y}{dx^3}+3P_1\frac{d^2y}{dx^2}+3P_2\frac{dy}{dx}+P_3y=0$$

may be expressed in the form

$$\frac{d^3y}{dx^3}+6Q_1\frac{d^2y}{dx^2}+2\left(\frac{dQ_1}{dx}+4Q_1^2+2Q_2\right)\frac{dy}{dx}+2\left(4Q_1Q_2+\frac{dQ_2}{dx}\right)y=0$$

therefore

$$P_1=2Q_1, \quad P_3=2\frac{dQ_2}{dx}+8Q_1Q_2$$

$$3P_2=2\frac{dQ_1}{dx}+8Q_1^2+4Q_2$$

From these equations eliminating $Q_1$, $Q_2$ we get

$$\frac{dL}{dx}+2P_1L-2P_3=0 \text{ or } K=0$$

as the required condition.

The relation $y_3^2=y_1y_2$ involving only the ratios of the solutions must be also expressible in terms of the invariants of the first kind considered in this paper, and in fact we find

$$\frac{dL}{dx}+2P_1L-2P_3=3\frac{dH}{dx}-2G$$

where H and G have the same meaning as before.

To arrive directly at this condition in terms of H and G, we see on referring to (B) that the cubic with the second term removed by the substitution for $y$ of $ye^{-\int P_1dx}$, viz.,

$$\frac{d^3y}{dx^3}+3H\frac{dy}{dx}+Gy=0$$

can be written in the form

$$\frac{d^3y}{dx^3}+4H_1\frac{dy}{dx}+2\frac{dH_1}{dx}y=0$$

where

$$H_1=Q_2-Q_1^2-\frac{dQ_1}{dx}$$

Therefore

$$3H=4H_1, \quad G=2\frac{dH_1}{dx}$$

and eliminating $H_1$ we find

$$3\frac{dH}{dx}-2G=0$$

the required result.

Hence the condition $y_3^2=y_1y_2$ may be expressed in either of the forms

$$K=0 \quad \text{or} \quad 3\frac{dH}{dx}-2G=0$$

we have in fact the relation

$$K^2=L^3\left(3\frac{dH}{dx}-2G\right)^2$$

We see also that in this case the solution of the cubic is reduced to that of the quadratic

$$\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+\frac{L}{4}y=0$$

the solutions of which are $\sqrt{y_1}$ and $\sqrt{y_2}$.

*The quartic.*

To find the invariants of the second kind of the equation of the fourth order

$$\frac{d^4y}{dx^4}+4P_1\frac{d^3y}{dx^3}+6P_2\frac{d^2y}{dx^2}+4P_3\frac{dy}{dx}+P_4y=0$$

let us suppose the second term removed by the substitution $x=\phi(t)$, then writing the result in the form

$$\frac{d^4y}{dt^4}+u\frac{d^2y}{dt^2}+v\frac{dy}{dt}+w=0$$

we find

$$2P_1\phi'-\frac{3\phi''}{\phi'}=0 \quad w=P_4\phi'^4$$

$$u=\frac{15\phi''^2}{\phi'^2}-\frac{4\phi'''}{\phi'}-12P_1\phi''+6P_2\phi'^2 \quad \ldots \ldots \quad (\alpha)$$

$$v=\frac{10\phi'''\phi''}{\phi'^2}-\frac{15\phi''^3}{\phi'^3}-\frac{\phi^{(iv)}}{\phi'}+12P_1\frac{\phi''^2}{\phi'}-4P_1\phi'''-6P_2\phi''\phi'+4P_3\phi'^3 \quad \ldots \quad (\beta)$$

From the first of these equations we have

$$2\frac{dP_1}{dx}=\frac{3\phi'''}{\phi'^3}-\frac{6\phi''^2}{\phi'^2}=\frac{3\phi'''}{\phi'^3}-\tfrac{8}{3}P_1^2$$

therefore by substitution in $(\alpha)$ we find

$$9u=\phi'^2\left(54P_2-44P_1^2-24\frac{dP_1}{dx}\right)$$

Hence we have

$$\frac{54P_2-44P_1^2-24\frac{dP_1}{dx}}{\sqrt{P_4}}=\frac{9u}{\sqrt{w}}$$

or

$$\frac{54P_2-44P_1^2-24\frac{dP_1}{dx}}{\sqrt{P_4}}=J_1 \text{ say}$$

is an invariant of the quartic of the kind we are considering.

Again by differentiating the equation

$$\frac{\phi'''}{\phi'^3}=\tfrac{2}{3}\frac{dP_1}{dx}+\tfrac{8}{9}P_1^2$$

we get

$$\frac{\phi^{(iv)}}{\phi'^4}=\tfrac{2}{3}\frac{d^2P_1}{dx^2}+\tfrac{16}{9}P_1\frac{dP_1}{dx}+\frac{3\phi'''\phi''}{\phi'^5}$$

$$=\tfrac{2}{3}\frac{d^2P_1}{dx^2}+\tfrac{16}{9}P_1\frac{dP_1}{dx}+2P_1\left(\tfrac{2}{3}\frac{dP_1}{dx}+\tfrac{8}{9}P_1^2\right)$$

$$=\tfrac{2}{3}\cdot\frac{d^2P_1}{dx^2}+\tfrac{28}{9}P_1\frac{dP_1}{dx}+\tfrac{16}{9}P_1^3$$

substituting in ($\beta$) we find

$$27v=\phi'^3\left\{40P_1^3-36P_1\frac{dP_1}{dx}-18\frac{d^2P_1}{dx^2}-108P_1P_2+108P_3\right\}$$

hence we see that

$$\frac{40P_1^3-36P_1\frac{dP_1}{dx}-18\frac{d^2P_1}{dx^2}-108P_1P_2+108P_3}{P_4^{\frac{3}{4}}}$$

is an invariant, which I shall call $J_2$.

We easily find a third invariant from the equations

$$2P_1=\frac{3\phi''}{\phi'^2} \quad \text{and} \quad w=P_4Q'^4,$$

namely,

$$\frac{3\frac{dP_4}{dx}+8P_1P_4}{P_4^{\frac{5}{4}}}$$

which may be called $J_3$.

By aid of the invariants $J_1$, $J_2$, $J_3$ we can solve problems with respect to the quartic which are analogous to those already treated of in the case of the cubic.

As an example, let the solutions be related by the equations

$$y_1 = y_2^m = y_3^n = y_4^p$$

then transforming so that $y_1$ shall be $e^t$ we find

$$J_1 = \text{ a constant, } J_2 = \text{ a constant, } J_3 = \text{ a constant ;}$$

and in this case the complete solution is

$$y = Ae^{\int\left(\frac{P_4}{\delta}\right)^{\frac{1}{4}}dx} + Be^{m\int\left(\frac{P_4}{\delta}\right)^{\frac{1}{4}}dx} + Ce^{n\int\left(\frac{P_4}{\delta}\right)^{\frac{1}{4}}dx} + De^{p\int\left(\frac{P_4}{\delta}\right)^{\frac{1}{4}}dx}$$

where

$$\delta = mnp$$